

6.3 – Gram-Schmidt Process

Definition: A set of two or more vectors in a real inner product space is said to be **orthogonal** if all pairs of distinct vectors in the set are orthogonal. An orthogonal set in which each vector has norm 1 is said to be **orthonormal**.

Theorem 6.3.1 If $S = \{v_1, v_2, \dots, v_n\}$ is an orthogonal set of nonzero vectors in an inner product space, then S is linearly independent.

Definition: In an inner product space, a basis comprising orthogonal vectors is an **orthogonal basis**, and a basis comprising orthonormal vectors is an **orthonormal basis**.

#6 Show that the column vectors of A form an orthogonal basis for the column space of A with respect to the Euclidean inner product, and then find an orthonormal basis for that column space.

$$A = \begin{bmatrix} 1/5 & -1/2 & 1/3 \\ 1/5 & 1/2 & 1/3 \\ 1/5 & 0 & -2/3 \end{bmatrix}$$

Theorem 6.3.2

a) If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is an orthogonal basis for an inner product space V , and if \mathbf{u} is any vector in V , then

$$\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 + \dots + \frac{\langle \mathbf{u}, \mathbf{v}_n \rangle}{\|\mathbf{v}_n\|^2} \mathbf{v}_n$$

b) If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is an orthonormal basis for an inner product space V , and if \mathbf{u} is any vector in V , then $\mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \dots + \langle \mathbf{u}, \mathbf{v}_n \rangle \mathbf{v}_n$.

#8 Use Theorem 6.3.2(b) to express the vector $\mathbf{u} = (3, -7, 4)$ as a linear combination of the vectors $\mathbf{v}_1 = \left(-\frac{3}{5}, \frac{4}{5}, 0\right)$, $\mathbf{v}_2 = \left(\frac{4}{5}, \frac{3}{5}, 0\right)$, $\mathbf{v}_3 = (0, 0, 1)$.

#12 Find the coordinate vector $(\mathbf{u})_S$ for the vector \mathbf{u} and the basis S that were given in Exercise 8.

Theorem 6.3.3 Projection Theorem (generalization of Theorem 3.3.2)

If W is a finite-dimensional subspace of an inner product space V , then every vector \mathbf{u} in V can be expressed in exactly one way as $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$ where \mathbf{w}_1 is in W and \mathbf{w}_2 is in W^\perp .

Definition: If W is a finite-dimensional subspace of an inner product space V and a vector \mathbf{u} in V is expressed as $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, where \mathbf{w}_1 is in W and \mathbf{w}_2 is in W^\perp , then \mathbf{w}_1 is called the **orthogonal projection of \mathbf{u} on W** and is denoted by $\mathbf{w}_1 = \text{proj}_W \mathbf{u}$ and \mathbf{w}_2 is called the **orthogonal projection of \mathbf{u} on W^\perp** and is denoted by $\mathbf{w}_2 = \text{proj}_{W^\perp} \mathbf{u}$. The vector \mathbf{w}_2 is also called the **component of \mathbf{u} orthogonal to W** .

Theorem 6.3.4 Let W be a finite-dimensional subspace of an inner product space V .

a) If $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ is an orthogonal basis for W and \mathbf{u} is any vector in V , then

$$\text{proj}_W \mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 + \dots + \frac{\langle \mathbf{u}, \mathbf{v}_r \rangle}{\|\mathbf{v}_r\|^2} \mathbf{v}_r$$

b) If $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ is an orthonormal basis for W and \mathbf{u} is any vector in V , then

$$\text{proj}_W \mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \dots + \langle \mathbf{u}, \mathbf{v}_r \rangle \mathbf{v}_r$$

#24 The vectors $\mathbf{v}_1 = (0, 1, -4, -1)$ and $\mathbf{v}_2 = (3, 5, 1, 1)$ are orthogonal with respect to the Euclidean inner product on R^4 . Find the orthogonal projection \mathbf{w}_1 of $\mathbf{b} = (1, 2, 0, -2)$ on the subspace W spanned by these vectors. [Extension of the exercise] Then find \mathbf{w}_2 in W^\perp such that $\mathbf{b} = \mathbf{w}_1 + \mathbf{w}_2$.

#25 The vectors $\mathbf{v}_1, \mathbf{v}_2,$ and \mathbf{v}_3 are orthonormal with respect to the Euclidean inner product on R^4 . Find the orthogonal projection of $\mathbf{b} = (1, 2, 0, -1)$ onto the subspace W spanned by these vectors.

$$\mathbf{v}_1 = \left(0, \frac{1}{\sqrt{18}}, -\frac{4}{\sqrt{18}}, -\frac{1}{\sqrt{18}}\right), \mathbf{v}_2 = \left(\frac{1}{2}, \frac{5}{6}, \frac{1}{6}, \frac{1}{6}\right), \mathbf{v}_3 = \left(\frac{1}{\sqrt{18}}, 0, \frac{1}{\sqrt{18}}, -\frac{4}{\sqrt{18}}\right)$$



Theorem 6.3.6 (inner product space analog of Theorem 4.6.5 (b))

If W is a finite-dimensional inner product space, then:

- a) Every orthogonal set of nonzero vectors in W can be enlarged to an orthogonal basis for W .
- b) Every orthonormal set of nonzero vectors in W can be enlarged to an orthonormal basis for W .

Example 6.3.9 Legendre Polynomials

Let the vector space P_2 have the inner product $\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^1 p(x)q(x)dx$. Apply the Gram-Schmidt process to transform the standard basis $\{1, x, x^2\}$ for P_2 into an orthogonal basis $\{\phi_1(x), \phi_2(x), \phi_3(x)\}$.
